CROSSED PRODUCTS OF LOCALLY C^*-ALGEBRAS AND MORITA EQUIVALENCE

MARIA JOIŢA

ABSTRACT. We introduce the notion of strong Morita equivalence for group actions on locally C^*-algebras and prove that the crossed products associated with two strongly Morita equivalent continuous inverse limit actions of a locally compact group G on the locally C^*-algebras A and B are strongly Morita equivalent. This generalizes a result of F. Combes, Proc. London Math. Soc. 49(1984) and R. E. Curto, P.S. Muhly, D. P. Williams, Proc. Amer. Soc. 90(1984).

1. Introduction

Locally C^*-algebras are generalizations of C^*-algebras. Instead of being given by a single C^*-norm, the topology on a locally C^*-algebra is defined by a directed family of C^*-seminorms. Such important concepts as Hilbert C^*-modules, Morita equivalence, crossed products of C^*-algebras can be defined in the framework of locally C^*-algebras and many results can be extended. The proofs, however, are not quite as straightforward.

Phillips [15] introduced the notion of inverse limit action of a locally compact group on a metrizable locally C^*-algebra and he considered the crossed products of metrizable locally C^*-algebras by an inverse limit action. In [8] we considered the crossed products of locally C^*-algebras by inverse limit actions and using some results about actions of Hopf C^*-algebras on locally C^*-algebras we proved a Takai duality theorem for crossed products of locally C^*-algebras.

The notion of strong Morita equivalence of locally C^*-algebras was introduced in [6]. It is well known that if α and β are two strongly Morita equivalent actions of a locally compact group G on two C^*-algebras A and B, then the crossed products $G \times_\alpha A$ and $G \times_\beta B$ are strongly Morita equivalent [2, 3]. In this work, we extend this result in the context of locally C^*-algebras.

The paper is organized as follows. In Section 2 are presented some results about locally C^*-algebras [4, 5, 6], Hilbert modules over locally C^*-algebras [10, 14] and crossed products of locally C^*-algebras [8, 15]. In Section 3, we introduce the notion of action of a locally compact group G on a Hilbert module E over a locally

2000 Mathematical Subject Classification: 46L08, 46L05, 46L55

This research was supported by CEEX grant-code PR-D11-PT00-48/2005 from The Romanian Ministry of Education and Research.

C^*-algebra A, and we show that, if E is full, such an action induces two actions of G on the locally C^*-algebras A and $K(E)$, the locally C^*-algebra of all compact operators on E. Moreover, these actions are inverse limit actions if the action of G on E is an inverse limit action. The notion of strong Morita equivalence for group actions on locally C^*-algebras is introduced in Section 4, and we prove that the strong Morita equivalence of group actions on locally C^*-algebras is an equivalence relation. In Section 5, we prove that the crossed products of two locally C^*-algebras by two strongly Morita equivalent inverse limit actions are strongly Morita equivalent.

2. Preliminaries

A locally C^*-algebra is a complete Hausdorff complex topological $*$-algebra A whose topology is determined by its continuous C^*-seminorms in the sense that a net $\{a_i\}_{i\in I}$ converges to 0 in A if and only if the net $\{p(a_i)\}_{i\in I}$ converges to 0 for all continuous C^*-seminorms p on A. The term of "locally C^*-algebra" is due to Inoue. In the literature, locally C^*-algebras have been given different names such as b^*-algebras (C. Apostol), LMC^*-algebras (G. Lassner, K. Schmüdgen) or pro-C^*-algebras (N. C. Phillips).

The set $S(A)$ of all continuous C^*-seminorms on A is directed with the order $p \geq q$ if $p\,(a) \geq q\,(a)$ for all $a \in A$. For each $p \in S(A)$, $\ker p = \{a \in A; p(a) = 0\}$ is a two-sided $*$-ideal of A and the quotient $*$-algebra $A/\ker p$, denoted by A_p, is a C^*-algebra in the C^*-norm induced by p. The canonical map from A to A_p is denoted by π_p^A.

For $p, q \in S(A)$ with $p \geq q$ there is a canonical surjective morphism of C^*-algebras $\pi_{pq}^A : A_p \to A_q$ such that $\pi_{pq}^A \circ \pi_p^A = \pi_q^A$. Then $\{A_p; \pi_{pq}^A\}_{p,q\in S(A),p\geq q}$ is an inverse system of C^*-algebras and moreover, the locally C^*-algebras A and $\lim\limits_{\substack{\leftarrow \\ p\in S(A)}} A_p$ are isomorphic.

A morphism of locally C^*-algebras is a continuous morphism of $*$ -algebras. Two locally C^*-algebras A and B are isomorphic if there is a bijective map $\Phi : A \to B$ such that Φ and Φ^{-1} are morphisms of locally C^*-algebras.

A representation of a locally C^*-algebra A on a Hilbert space H is a continuous $*$-morphism φ from A to $L(H)$, the C^*-algebra of all bounded linear operators on H. We say that the representation φ is non-degenerate if $\varphi\,(A)\,H$ is dense in H.

Hilbert modules over locally C^*-algebras are generalizations of Hilbert C^*-modules by allowing the inner product to take values in a locally C^*-algebra rather than in a C^*-algebra.

Definition 2.1. *A pre-Hilbert A-module is a complex vector space E which is also a right A-module, compatible with the complex algebra structure, equipped with an A-valued inner product $\langle\cdot,\cdot\rangle : E \times E \to A$ which is $\mathbb{C}$ -and A-linear in its second variable and satisfies the following relations:*

(1) $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ *for every* $\xi, \eta \in E$;
(2) $\langle \xi, \xi \rangle \geq 0$ *for every* $\xi \in E$;
(3) $\langle \xi, \xi \rangle \geq 0$ *for every* $\xi \in E; \langle \xi, \xi \rangle = 0$ *if and only if* $\xi = 0$.

We say that E *is a Hilbert* A*-module if* E *is complete with respect to the topology determined by the family of seminorms* $\{\overline{p}_E\}_{p \in S(A)}$, *where* $\overline{p}_E(\xi) = \sqrt{p(\langle \xi, \xi \rangle)}$, $\xi \in E$.

Any locally C^*-algebra A is a Hilbert A -module in a natural way.

A Hilbert A-module E is full if the linear space $\langle E, E \rangle$ generated by $\{\langle \xi, \eta \rangle, \ \xi, \eta \in E\}$ is dense in A.

Let E be a Hilbert A-module. For $p \in S(A)$, $\ker \overline{p}_E = \{\xi \in E; \overline{p}_E(\xi) = 0\}$ is a closed submodule of E and $E_p = E / \ker \overline{p}_E$ is a Hilbert A_p-module with $(\xi + \ker \overline{p}_E)\pi_p^A(a) = \xi a + \ker \overline{p}_E$ and $\langle \xi + \ker \overline{p}_E, \eta + \ker \overline{p}_E \rangle = \pi_p^A(\langle \xi, \eta \rangle)$. The canonical map from E onto E_p is denoted by σ_p^E.

For $p, q \in S(A)$ with $p \geq q$, there is a canonical surjective morphism of vector spaces σ_{pq}^E from E_p onto E_q such that $\sigma_{pq}^E(\sigma_p^E(\xi)) = \sigma_q^E(\xi)$ for all $\xi \in E$. Then $\{E_p; A_p; \sigma_{pq}^E, \ \pi_{pq}^A\}_{p,q \in S(A), p \geq q}$ is an inverse system of Hilbert C^*-modules in the following sense: $\sigma_{pq}^E(\xi_p a_p) = \sigma_{pq}^E(\xi_p)\pi_{pq}^A(a_p), \xi_p \in E_p, a_p \in A_p$; $\left\langle \sigma_{pq}^E(\xi_p), \sigma_{pq}^E(\eta_p) \right\rangle = \pi_{pq}^A(\langle \xi_p, \eta_p \rangle), \xi_p, \ \eta_p \in E_p$; $\sigma_{pp}^E(\xi_p) = \xi_p, \ \xi_p \in E_p$ and $\sigma_{qr}^E \circ \sigma_{pq}^E = \sigma_{pr}^E$ if $p \geq q \geq r$, and $\lim\limits_{\substack{\longleftarrow \\ p \in S(A)}} E_p$ is a Hilbert A-module with the action defined by $\left((\xi_p)_p\right)((a_p)_p) = (\xi_p a_p)_p$ and the inner product defined by $\left\langle (\xi_p)_p, (\eta_p)_p \right\rangle = \left(\langle \xi_p, \eta_p \rangle\right)_p$. Moreover, the Hilbert A-module E can be identified with $\lim\limits_{\substack{\longleftarrow \\ p \in S(A)}} E_p$.

Let E and F be two Hilbert A-modules. A module morphism $T : E \to F$ is adjointable if there is a module morphism $T^* : F \to E$ such that $\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle$ for all $\xi \in E$ and for all $\eta \in F$. If T is an adjointable module morphism from E to F, then, for each $p \in S(A)$, there is a positive constant M_p such that $\overline{p}_F(T\xi) \leq M_p \overline{p}_E(\xi)$ for all $\xi \in E$.

The set of all adjointable module morphisms from E to F is denoted by $L(E, F)$ and we write $L(E)$ for $L(E, E)$. For $p \in S(A)$, since $T(\ker \overline{p}_E) \subseteq \ker \overline{p}_F$ for all $T \in L(E, F)$, we can define a linear map $(\pi_p^A)_* : L(E, F) \to L(E_p, F_p)$ by

$$(\pi_p^A)_*(T)(\sigma_p^E(\xi)) = \sigma_p^F(T(\xi)).$$

We consider on $L(E, F)$ the topology defined by family of seminorms $\{\widetilde{p}_{L(E,F)}\}_{p \in S(A)}$, where

$$\widetilde{p}_{L(E,F)}(T) = \left\|(\pi_p^A)_*(T)\right\|_{L(E_p, F_p)}$$

for all $T \in L(E, F)$. Thus topologized $L(E, F)$ becomes a complete locally convex space and $L(E)$ becomes a locally C^*-algebra.

For $p,q \in S(A)$ with $p \geq q$, consider the linear map $(\pi_{pq}^{A})_{*} : L(E_p, F_p) \rightarrow L(E_q, F_q)$ defined by

$$(\pi_{pq}^{A})_{*}(T_p)(\sigma_q^E(\xi)) = \sigma_{pq}^F(T_p(\sigma_p^E(\xi)))$$

$T_p \in L(E_p, F_p),\ \xi \in E$. Then $\{L(E_p, F_p), (\pi_{pq}^{A})_{*}\}_{p,q \in S(A), p \geq q}$ is an inverse system of Banach spaces and the complete locally convex spaces $L(E,F)$ and $\lim\limits_{\underset{p\in S(A)}{\leftarrow}} L(E_p, F_p)$ can be identified. Moreover, the locally C^*-algebras $L(E)$ and $\lim\limits_{\underset{p\in S(A)}{\leftarrow}} L(E_p)$ can be identified.

For $\xi \in E$ and $\eta \in F$ we consider the rank one homomorphism $\theta_{\eta,\xi}$ from E into F defined by $\theta_{\eta,\xi}(\zeta) = \eta \langle \xi, \zeta \rangle$. Clearly, $\theta_{\eta,\xi} \in L(E,F)$ and $\theta_{\eta,\xi}^{*} = \theta_{\xi,\eta}$. The linear subspace of $L(E,F)$ spanned by $\{\theta_{\eta,\xi}; \xi \in E, \eta \in F\}$ is denoted by $\Theta(E,F)$, and the closure of $\Theta(E,F)$ in $L(E,F)$ is denoted by $K(E,F)$. We write $K(E)$ for $K(E,E)$. Moreover, $K(E,F)$ may be identified with $\lim\limits_{\underset{p\in S(A)}{\leftarrow}} K(E_p, F_p)$.

We say that the Hilbert A-modules E and F are unitarily equivalent if there is a unitary element U in $L(E,F)$ (that is, $U^*U = \mathrm{id}_E$ and $UU^* = \mathrm{id}_F$).

Let G be a locally compact group and let A be a locally C^*-algebra. The vector space of all continuous functions from G to A with compact support is denoted by $C_c(G,A)$.

Lemma 2.2. *([8, Lemma 3.7]]) Let $f \in C_c(G,A)$. Then there is a unique element $\int\limits_G f(s)ds$ in A such that for any non-degenerate $*$-representation (φ, H_φ) of A*

$$\left\langle \varphi(\int\limits_G f(s)ds)\xi, \eta \right\rangle = \int\limits_G \langle \varphi(f(s))\xi, \eta \rangle \, ds$$

for all ξ, η in H_φ. Moreover, we have:

(1) $p(\int\limits_G f(s)ds) \leq M_f \sup\{p(f(s)); s \in supp(f)\}$ for some positive constant M_f and for all $p \in S(A)$;

(2) $(\int\limits_G f(s)ds)a = \int\limits_G f(s)ads$ for all $a \in A$;

(3) $\Phi(\int\limits_G f(s)ds) = \int\limits_G \Phi(f(s))\,ds$ for any morphism of locally C^*-algebras $\Phi : A \rightarrow B$;

(4) $(\int\limits_G f(s)ds)^* = \int\limits_G f(s)^* ds$.

An action of G on A is a morphism of groups α from G to $\mathrm{Aut}(A)$, the set of all isomorphisms of locally C^*-algebras from A to A. The action α is continuous if the function $t \rightarrow \alpha_t(a)$ from G to A is continuous for each $a \in A$.

An action α of G on A is an inverse limit action if we can write A as inverse limit $\lim\limits_{\substack{\leftarrow \\ \lambda\in\Lambda}} A_\lambda$ of C^*-algebras in such a way that there are actions α^λ of G on A_λ, $\lambda \in \Lambda$ such that $\alpha_t = \lim\limits_{\substack{\leftarrow \\ \lambda\in\Lambda}} \alpha_t^\lambda$ for all t in G [15, Definition 5.1].

The action α of G on A is a continuous inverse limit action if there is a cofinal subset of G-invariant continuous C^*-seminorms on A (a continuous C^*-seminorm p on A is G -invariant if $p(\alpha_t(a)) = p(a)$ for all a in A and for all t in G). Thus, for a continuous inverse limit action $t \to \alpha_t$ of G on A we can suppose that for each $p \in S(A)$, there is a continuous action $t \to \alpha_t^p$ of G on A_p such that $\alpha_t = \lim\limits_{\substack{\leftarrow \\ p\in S(A)}} \alpha_t^p$ for all $t \in G$.

Let $f, h \in C_c(G, A)$. The map $(s,t) \to f(t)\alpha_t\left(h(t^{-1}s)\right)$ from $G \times G$ to A is an element in $C_c(G \times G, A)$ and the relation

$$(f \times h)(s) = \int\limits_G f(t)\alpha_t\left(h(t^{-1}s)\right) dt$$

defines an element in $C_c(G, A)$, called the convolution of f and h.

The vector space $C_c(G, A)$ becomes a $*$-algebra with convolution as product and involution defined by

$$f^\sharp(t) = \Delta(t)^{-1}\alpha_t\left(f(t^{-1})^*\right)$$

where Δ is the modular function on G.

For any $p \in S(A)$, the map $N_p : C_c(G, A) \to [0, \infty)$ defined by

$$N_p(f) = \int\limits_G p(f(s))ds$$

is a submultiplicative $*$-seminorm on $C_c(G, A)$.

Let $L^1(G, A, \alpha)$ be the Hausdorff completion of $C_c(G, A)$ with respect to the topology defined by the family of submultiplicative $*$-seminorms $\{N_p\}_{p\in S(A)}$. Then $L^1(G, A, \alpha)$ is a complete locally m-convex $*$-algebra.

For each $p \in S(A)$ the map $n_p : L^1(G, A, \alpha) \to [0, \infty)$ defined by

$$n_p(f) = \sup\{\|\varphi(f)\|\,; \varphi \in \mathcal{R}_p(L^1(G, A, \alpha)\},$$

where $\mathcal{R}_p(L^1(G, A, \alpha)$ denotes the set of all non-degenerate representations φ of $L^1(G, A, \alpha)$ on Hilbert spaces which verify the relation

$$\|\varphi(f)\| \leq N_p(f)$$

for all $f \in L^1(G, A, \alpha)$, is a C^*-seminorm on $L^1(G, A, \alpha)$. The Hausdorff completion of $L^1(G, A, \alpha)$ with respect to the topology determined by the family of C^*-seminorms $\{n_p\}_{p\in S(A)}$ is a locally C^*-algebra, denoted by $G \times_\alpha A$, and called the crossed product of A by the action α. Moreover,

$$A \times_\alpha G = \lim_{\substack{\leftarrow \\ p\in S(A)}} G \times_{\alpha^p} A_p$$

up to an isomorphism of locally C^*-algebras.

3. Group actions on Hilbert modules

Let A and B be two locally C^*-algebras, let E be a Hilbert A-module and let F be a Hilbert B-module.

Definition 3.1. *A map $u : E \to F$ is called morphism of Hilbert modules if there is a morphism of locally C^*-algebras $\alpha : A \to B$ such that*

$$\langle u(\xi), u(\eta)\rangle = \alpha(\langle \xi, \eta\rangle)$$

for all $\xi, \eta \in E$. If we want to specify the morphism α of locally C^*-algebras, we say that u is an α-morphism.

Remark 3.2. *Let $u : E \to F$ be an α-morphism. Then:*

(1) *u is linear;*
(2) *u is continuous;*
(3) *$u(\xi a) = u(\xi)\alpha(a)$ for all $a \in A$ and for all $\xi \in E$;*
(4) *if α is injective, then u is injective.*

Definition 3.3. *An isomorphism of Hilbert modules is a bijective map $u : E \to F$ such that u and u^{-1} are morphisms of Hilbert modules.*

Proposition 3.4. *Let E be a Hilbert A-module, let F be a Hilbert B-module and let $u : E \to F$ be an α-morphism of Hilbert modules. If E and F are full and u is an isomorphism of Hilbert modules, then α is an isomorphism of locally C^*-algebras.*

Proof. Indeed, since $u^{-1} : F \to E$ is a morphism of Hilbert modules, there is a morphism of locally C^*-algebras $\beta : B \to A$ such that

$$\beta(\langle \eta_1, \eta_2\rangle) = \left\langle u^{-1}(\eta_1), u^{-1}(\eta_2)\right\rangle$$

for all $\eta_1, \eta_2 \in F$. Then we have:

$$(\alpha \circ \beta)(\langle \eta_1, \eta_2\rangle) = \alpha\left(\left\langle u^{-1}(\eta_1), u^{-1}(\eta_2)\right\rangle\right) = \langle \eta_1, \eta_2\rangle$$

for all $\eta_1, \eta_2 \in F$ and

$$(\beta \circ \alpha)(\langle \xi_1, \xi_2\rangle) = \beta(\langle u(\xi_1), u(\xi_2)\rangle) = \langle \xi_1, \xi_2\rangle$$

for all $\xi_1, \xi_2 \in E$. From these facts and taking into account that E and F are full and the maps α and β are continuous, we conclude that $\alpha \circ \beta = \mathrm{id}_B$ and $\beta \circ \alpha = \mathrm{id}_A$. Therefore α is an isomorphism of locally C^*-algebras. $\square$

For a Hilbert A-module E,

$$\mathrm{Aut}(E) = \{u : E \to E; u \text{ is an isomorphism of Hilbert modules }\}$$

is a group.

Definition 3.5. *Let G be a locally compact group. An action of G on E is a morphism of groups $g \to u_g$ from G to $Aut(E)$.*

The action $g \to u_g$ of G on E is continuous if the map $g \to u_g(\xi)$ from G to E is continuous for each $\xi \in E$.

An action $g \to u_g$ of G on E is an inverse limit action if we can write E as an inverse limit $\lim\limits_{\stackrel{\leftarrow}{\lambda \in \Lambda}} E_\lambda$ of Hilbert C^*-modules in such a way that there are actions $g \to u_g^\lambda$ of G on E_λ, $\lambda \in \Lambda$ such that $u_g = \lim\limits_{\stackrel{\leftarrow}{\lambda \in \Lambda}} u_g^\lambda$ for each $g \in G$.

Remark 3.6. *Suppose that $g \to u_g$ is an inverse limit action of G on E. Then $E = \lim\limits_{\stackrel{\leftarrow}{\lambda \in \Lambda}} E_\lambda$ and $u_g = \lim\limits_{\stackrel{\leftarrow}{\lambda \in \Lambda}} u_g^\lambda$ for each $g \in G$, where $g \to u_g^\lambda$ is an action of G on E_λ for each $\lambda \in \Lambda$.*

Let $\lambda \in \Lambda$. Since $g \to u_g^\lambda$ is an action of G on E_λ,

$$\left\| u_g^\lambda \left(\sigma_\lambda(\xi)\right) \right\|_{E_\lambda} = \left\| \sigma_\lambda(\xi \right\|_{E_\lambda}$$

for each $\xi \in E$, and for all $g \in G$ [2, pp. 292]*. This implies that*

$$\overline{p}_\lambda(u_g(\xi)) = \overline{p}_\lambda(\xi)$$

for all $g \in G$ and for all $\xi \in E$.

From these facts, we conclude that $g \to u_g$ is an inverse limit action of G on E, if $S(G, A) = \{p \in S(A); \overline{p}_E\left(u_g(\xi)\right) = \overline{p}_E(\xi)$ for all $g \in G$ and for all $\xi \in E\}$ is a cofinal subset of $S(A)$. Therefore, if $g \to u_g$ is an inverse limit action of G on E, we can suppose that $u_g = \lim\limits_{\stackrel{\leftarrow}{p \in S(A)}} u_g^p$.

Remark 3.7. *Let $g \to u_g$ be an inverse limit action of G on E. By Remark 3.6, we can suppose that $u_g = \lim\limits_{\stackrel{\leftarrow}{p \in S(A)}} u_g^p$. If the actions $g \to u_g^p$ of G on E_p, $p \in S(A)$ are all continuous, then, clearly, the action $g \to u_g$ of G on E is continuous.*

Conversely, suppose that the action $g \to u_g$ of G on E is continuous. Let $p \in S(A)$, $g_0 \in G$, $\xi_0 \in E$ and $\varepsilon > 0$. Since the map $g \to u_g(\xi_0)$ from G to E is continuous, there is a neighborhood U_0 of g_0 such that

$$\overline{p}_E\left(u_g(\xi_0) - u_{g_0}(\xi_0)\right) \leq \varepsilon$$

for all $g \in U_0$. Then

$$\begin{aligned} \left\| u_g^p\left(\sigma_p^E(\xi_0)\right) - u_{g_0}^p\left(\sigma_p^E(\xi_0)\right) \right\|_{E_p} &= \left\| \sigma_p^E\left(u_g(\xi_0)\right) - \sigma_p^E\left(u_{g_0}(\xi_0)\right) \right\|_{E_p} \\ &= \overline{p}_E\left(u_g(\xi_0) - u_{g_0}(\xi_0)\right) \leq \varepsilon \end{aligned}$$

for all $g \in U_0$. This shows that the action $g \to u_g^p$ of G on E_p is continuous. Thus we showed that the inverse limit action $g \to u_g$ of G on E is continuous if and only if the actions $g \to u_g^p$ of G on E_p, $p \in S(A)$ are all continuous.

Proposition 3.8. *Let G be a locally compact group and let E be a full Hilbert A-module. Any action $g \to u_g$ of G on E induces an action $g \to \alpha_g^u$ of G on A such that*

$$\alpha_g^u(\langle \xi, \eta \rangle) = \langle u_g(\xi), u_g(\eta) \rangle$$

for all $g \in G$ and for all $\xi, \eta \in E$ and an action $g \to \beta_g^u$ of G on $K(E)$ such that

$$\beta_g^u(\theta_{\xi,\eta}) = \theta_{u_g(\xi), u_g(\eta)}$$

for all $g \in G$ and for all $\xi, \eta \in E$. Moreover, if $g \to u_g$ is a continuous inverse limit action of G on E, then the actions of G on A respectively $K(E)$ induced by u are continuous inverse limit actions.

Proof. Let $g \in G$. Since E is full and $u_g \in$Aut(E), there is an isomorphism of locally C^*-algebras $\alpha_g^u : A \to A$ such that

$$\alpha_g^u(\langle \xi, \eta \rangle) = \langle u_g(\xi), u_g(\eta) \rangle$$

for all $\xi, \eta \in E$. It is not difficult to check that the map $g \to \alpha_g^u$ from G to Aut(A) is a morphism of groups. Therefore $g \to \alpha_g^u$ is an action of G on A.

Let $g \in G$. Consider the linear map $\beta_g^u : \Theta(E) \to \Theta(E)$ defined by

$$\beta_g^u(\theta_{\xi,\eta}) = \theta_{u_g(\xi), u_g(\eta)}.$$

It is not difficult to check that β_g^u is a $*$-morphism.

Let $p \in S(A)$. Then

$$\begin{aligned} \widetilde{p}_{L(E)}\left(\beta_g^u(\theta_{\xi,\eta})\right) &= \sup\{\overline{p}_E(u_g(\xi)\langle u_g(\eta), u_g(\zeta)\rangle); \overline{p}_E(u_g(\zeta)) \leq 1\} \\ &= \sup\{\overline{p}_E(u_g(\xi\langle \eta, \zeta\rangle)); \overline{p}_E(u_g(\zeta)) \leq 1\} \\ &= \sup\{\left(p \circ \alpha_g^u\right)(\langle \zeta, \eta\rangle\langle \xi, \xi\rangle\langle \eta, \zeta\rangle)^{1/2}; \left(p \circ \alpha_g^u\right)(\zeta) \leq 1\} \\ &= \sup\{r(\langle \zeta, \eta\rangle\langle \xi, \xi\rangle\langle \eta, \zeta\rangle)^{1/2}; r(\zeta) \leq 1\} \\ &= \widetilde{r}_{L(E)}(\theta_{\xi,\eta}) \end{aligned}$$

where $r = p \circ \alpha_g^u \in S(A)$, for all $\xi, \eta \in E$. From this fact, we conclude that β_g^u extends to a morphism from $K(E)$ to $K(E)$, denoted also by β_g^u. Moreover, since

$$\left(\beta_g^u \circ \beta_{g^{-1}}^u\right)(\theta_{\xi,\eta}) = \theta_{\xi,\eta} = \left(\beta_{g^{-1}}^u \circ \beta_g^u\right)(\theta_{\xi,\eta})$$

for all $\xi, \eta \in E$, β_g^u is invertible and $\left(\beta_g^u\right)^{-1} = \beta_{g^{-1}}^u$. Therefore $\beta_g^u \in$Aut$(K(E))$. It is not difficult to check that the map $g \to \beta_g^u$ is a morphism of groups and so it defines an action of G on $K(E)$.

Now suppose that $g \to u_g$ is a continuous inverse limit action of G on E. Let $p \in S(A)$. Then the map $g \to u_g^p$ is a continuous action of G on E_p and so it induces a continuous action $g \to \alpha_g^{u^p}$ of G on A_p such that

$$\alpha_g^{u^p}\left(\langle \sigma_p^E(\xi), \sigma_p^E(\eta)\rangle\right) = \left\langle u_g^p\left(\sigma_p^E(\xi)\right), u_g^p\left(\sigma_p^E(\eta)\right)\right\rangle$$

for all $\xi, \eta \in E$ and for all $g \in G$ and a continuous action $g \to \beta_g^{u^p}$ of G on $K(E_p)$ such that

$$\beta_g^{u^p}\left(\theta_{\sigma_p^E(\xi),\sigma_p^E(\eta)}\right) = \theta_{u_g^p\left(\sigma_p^E(\xi)\right),u_g^p\left(\sigma_p^E(\eta)\right)}$$

for all $g \in G$ and for all $\xi, \eta \in E$ (see, for example, [2]).

Let $p, q \in S(A)$ with $p \geq q$ and $g \in G$. Then:

$$\begin{aligned}
\left(\pi_{pq}^A \circ \alpha_g^{u^p}\right)\left(\left\langle \sigma_p^E(\xi), \sigma_p^E(\eta)\right\rangle\right) &= \pi_{pq}^A\left(\left\langle u_g^p\left(\sigma_p^E(\xi)\right), u_g^p\left(\sigma_p^E(\eta)\right)\right\rangle\right) \\
&= \left\langle \left(\sigma_{pq}^E \circ u_g^p\right)\left(\sigma_p^E(\xi)\right), \left(\sigma_{pq}^E \circ u_g^p\right)\left(\sigma_p^E(\eta)\right)\right\rangle \\
&= \left\langle u_g^q\left(\sigma_q^E(\xi)\right), u_g^q\left(\sigma_q^E(\eta)\right)\right\rangle \\
&= \alpha_g^{u^q}\left(\left\langle \sigma_q^E(\xi), \sigma_q^E(\eta)\right\rangle\right) \\
&= \left(\alpha_g^{u^q} \circ \pi_{pq}^A\right)\left(\left\langle \sigma_p^E(\xi), \sigma_p^E(\eta)\right\rangle\right)
\end{aligned}$$

and

$$\begin{aligned}
\left(\left(\pi_{pq}^A\right)_* \circ \beta_g^{u^p}\right)\left(\theta_{\sigma_p^E(\xi),\sigma_p^E(\eta)}\right) &= \left(\pi_{pq}^A\right)_*\left(\theta_{u_g^p\left(\sigma_p^E(\xi)\right),u_g^p\left(\sigma_p^E(\eta)\right)}\right) \\
&= \theta_{\left(\sigma_{pq}^E \circ u_g^p\right)\left(\sigma_p^E(\xi)\right),\left(\sigma_{pq}^E \circ u_g^p\right)\left(\sigma_p^E(\eta)\right)} \\
&= \theta_{u_g^q\left(\sigma_q^E(\xi)\right),u_g^q\left(\sigma_q^E(\eta)\right)} \\
&= \beta_g^{u^q}\left(\theta_{\sigma_q^E(\xi),\sigma_q^E(\eta)}\right) \\
&= \left(\beta_g^{u^q} \circ \left(\pi_{pq}^A\right)_*\right)\left(\theta_{\sigma_p^E(\xi),\sigma_p^E(\eta)}\right)
\end{aligned}$$

for all $\xi, \eta \in E$. From these relations, we conclude that $(\alpha_g^{u^p})_p$ and $\left(\beta_g^{u^p}\right)_p$ are inverse systems of C^*-isomorphisms, and then $\lim\limits_{\underset{p\in S(A)}{\leftarrow}} \alpha_g^{u^p}$ and $\lim\limits_{\underset{p\in S(A)}{\leftarrow}} \beta_g^{u^p}$ are isomorphisms of locally C^*-algebras. It is not difficult to check that $g \to \lim\limits_{\underset{p\in S(A)}{\leftarrow}} \alpha_g^{u^p}$ and $g \to \lim\limits_{\underset{p\in S(A)}{\leftarrow}} \beta_g^{u^p}$ are continuous inverse limit actions of G on A respectively $K(E)$. Moreover, $\alpha_g^u = \lim\limits_{\underset{p\in S(A)}{\leftarrow}} \alpha_g^{u^p}$ of G and $\beta_g^u = \lim\limits_{\underset{p\in S(A)}{\leftarrow}} \beta_g^{u^p}$ for all $g \in G$. □

4. Morita equivalence of group actions on locally C^*-algebras

In this section we introduce the notion of (strong) Morita equivalence of group actions on locally C^*-algebras and show that this is an equivalence relation.

Definition 4.1. *Let G be a locally compact group, let A and B be two locally C^*-algebras. Two actions $g \to \alpha_g$ and $g \to \beta_g$ of G on A respectively B are conjugate if there is an isomorphism of locally C^*-algebras $\varphi : A \to B$ such that $\alpha_g = \varphi^{-1} \circ \beta_g \circ \varphi$ for each $g \in G$.*

Remark 4.2. *Conjugacy of group actions on locally C^*-algebras is an equivalence relation.*

Lemma 4.3. *Let $g \to \alpha_g$ and $g \to \beta_g$ be two conjugate actions of G on the locally C^*-algebras A respectively B. If $g \to \alpha_g$ is a continuous inverse limit action of G on A, then $g \to \beta_g$ is a continuous inverse limit action of G on B.*

Proof. Suppose that $\alpha_g = \lim\limits_{\substack{\leftarrow \\ p\in S(A)}} \alpha_g^p$ for each $g \in G$. Let $\varphi : A \to B$ be an isomorphism of locally C^*-algebras such that $\alpha_g = \varphi^{-1} \circ \beta_g \circ \varphi$ for each $g \in G$.

Let $p \in S(A)$. Then $p \circ \varphi^{-1}$ is a continuous C^*-seminorm on B. We show that $\{p \circ \varphi^{-1}\}_{p\in S(A)}$ is a cofinal subset of $S(B)$. For this, let $q \in S(B)$. Since φ is an isomorphism of locally C^*-algebras, there is $p \in S(A)$ such that

$$q\left(\varphi\left(a\right)\right) \leq p(a)$$

for all $a \in A$. Then

$$q\left(b\right) = q\left(\varphi\left(\varphi^{-1}(b)\right)\right) \leq p\left(\varphi^{-1}(b)\right) = \left(p \circ \varphi^{-1}\right)(b)$$

for all $b \in B$. Therefore $\{p \circ \varphi^{-1}\}_{p\in S(A)}$ is a cofinal subset of $S(B)$ and thus we can identify B with $\lim\limits_{\substack{\leftarrow \\ p\in S(A)}} B_{p\circ\varphi^{-1}}$.

For each $p \in S(A)$, there is an isomorphism of C^*-algebras $\varphi_p : A_p \to B_{p\circ\varphi^{-1}}$ such that

$$\varphi_p \circ \pi_p^A = \pi_{p\circ\varphi^{-1}}^B \circ \varphi$$

since $\ker \pi_p^A = \ker\left(\pi_{p\circ\varphi^{-1}}^B \circ \varphi\right)$. The map $g \to \beta_g^p$, where $\beta_g^p = \varphi_p \circ \alpha_g^p \circ \varphi_p^{-1}$, is a continuous action of G on $B_{p\circ\varphi^{-1}}$ which is conjugate with α^p. It is not difficult to check that $(\beta_g^p)_p$ is an inverse system of C^*-isomorphisms and $\beta_g = \lim\limits_{\substack{\leftarrow \\ p\in S(A)}} \beta_g^p$ for each $g \in G$. Therefore $g \to \beta_g$ is a continuous inverse limit action of G on B. □

Proposition 4.4. *Let $g \to \alpha_g$ and $g \to \beta_g$ be two continuous inverse limit actions of G on the locally C^*-algebras A and B. If the actions α and β are conjugate, then the crossed products $G \times_\alpha A$ and $G \times_\beta B$ are isomorphic.*

Proof. By Lemma 4.3, we can suppose that $S(B) = \{p \circ \varphi^{-1}\}_{p\in S(A)}$, where φ is an isomorphism of locally C^*-algebras from A to B such that $\beta_g = \varphi \circ \alpha_g \circ \varphi^{-1}$. Then $B = \lim\limits_{\substack{\leftarrow \\ p\in S(A)}} B_{p\circ\varphi^{-1}}$, $\alpha_g = \lim\limits_{\substack{\leftarrow \\ p\in S(A)}} \alpha_g^p$ for each $g \in G$ and $\beta_g = \lim\limits_{\substack{\leftarrow \\ p\in S(A)}} \beta_g^p$ for each $g \in G$. Moreover, for each $p \in S(A)$, the actions $g \to \alpha_g^p$ of G on A_p and $g \to \beta_g^p$ of G on $B_{p\circ\varphi^{-1}}$ are conjugate.

Let $p \in S(A)$, and let φ_p be the isomorphism of C^*-algebras from A_p to $B_{p\circ\varphi^{-1}}$ such that $\beta_g^p = \varphi_p \circ \alpha_g^p \circ \varphi_p^{-1}$. Then the linear map $\phi_p : C_c(G, A_p) \to C_c(G, B_{p\circ\varphi^{-1}})$ defined by

$$\phi_p\left(f_p\right) = \varphi_p \circ f_p$$

extends to an isomorphism of C^*-algebras from $G\times_{\alpha^p} A_p$ to $G\times_{\beta^p} B_{p\circ\varphi^{-1}}$ [17, pp. 66]. Since

$$\begin{aligned}\left(\pi_{pq}^{G\times_\beta B}\circ\phi_p\right)(f_p) &= \pi_{pq}^{B}\circ\phi_p(f_p)=\pi_{pq}^{B}\circ\varphi_p\circ f_p\\ &= \left(\varphi_q\circ\pi_{pq}^{A}\right)\circ f_p=\varphi_q\circ\left(\pi_{pq}^{A}\circ f_p\right)\\ &= \phi_q\left(\pi_{pq}^{G\times_\alpha A}(f_p)\right)=\left(\phi_q\circ\pi_{pq}^{G\times_\alpha A}\right)(f_p)\end{aligned}$$

for all $f_p\in C_c(G,A_p)$ and for all $p,q\in S(A)$ with $p\geq q$, $\left(\phi_p\right)_p$ is an inverse system of isomorphisms of C^*-algebras and then $\phi=\lim\limits_{\substack{\leftarrow\\ p\in S(A)}}\phi_p$ is an isomorphism of locally C^*-algebras from $\lim\limits_{\substack{\leftarrow\\ p\in S(A)}} G\times_{\alpha^p} A_p$ to $\lim\limits_{\substack{\leftarrow\\ p\in S(A)}} G\times_{\beta^p} B_p$. Therefore the locally C^*-algebras $G\times_\alpha A$ and $G\times_\beta B$ are isomorphic. □

Definition 4.5. *Let G be a locally compact group, let A and B be two locally C^*-algebras, and let $g\to\alpha_g$ and $g\to\beta_g$ be two actions of G on A and B. We say that α and β are (strongly) Morita equivalent, if there exist a full Hilbert A-module E, and a (continuous) action $g\to u_g$ of G on E such that the actions of G on A and $K(E)$ induced by u are conjugate with the actions α respectively β. We write $\alpha\sim_{E,u}\beta$ if the actions α and β are Morita equivalent and $\alpha\sim_{E,u}^{s}\beta$ if the actions α and β are strongly Morita equivalent.*

Remark 4.6. *Two conjugate (continuous inverse limit) actions of G on the locally C^*-algebras A and B are (strongly) Morita equivalent.*

Remark 4.7. *If $\alpha\sim_{E,u}^{s}\beta$ and α is a continuous inverse limit action, then u and β are continuous inverse limit actions. Indeed, for any $p\in S(A)$ we have*

$$\overline{p}_E(u_g(\xi))^2=p\left(\langle u_g(\xi),u_g(\xi)\rangle\right)=p\left(\alpha_g\left(\langle\xi,\xi\rangle\right)\right)=\overline{p}_E(\xi)^2$$

for all $g\in G$ and for all $\xi\in E$, and then, by Remark 3.6, $g\to u_g$ is a continuous inverse limit action of G on E. Moreover, by Proposition 3.8 and Lemma 4.3, $g\to\beta_g$ is a continuous inverse limit action.

Let E be a full Hilbert A-module and let $\widetilde{E}=K(E,A)$. Then $\widetilde{E}$ is a Hilbert $K(E)$-module in a natural way [6, pp. 805-806]. Moreover, $\widetilde{E}$ is full and the locally C^*-algebras $K(\widetilde{E})$ and A are isomorphic [6, pp. 805-806].

Proposition 4.8. *Let G be a locally compact group and let $g\to u_g$ be an action of G on a full Hilbert A-module E. Then u induces an action $g\to\widetilde{u}_g$ of G on the Hilbert $K(E)$-module $\widetilde{E}$ such that $\widetilde{u}_g(\theta_{a,\xi})=\theta_{\alpha_g^u(a),u_g(\xi)}$ for all $a\in A$ and for all $\xi\in E$. Moreover, if u is a continuous inverse limit action, then $\widetilde{u}$ is a continuous inverse limit action.*

Proof. Let $g\in G$. Consider the linear map $\widetilde{u}_g:\Theta(E,A)\to\Theta(E,A)$ defined by

$$\widetilde{u}_g(\theta_{a,\xi})=\theta_{\alpha_g^u(a),u_g(\xi)}.$$

Since

$$\begin{aligned}\overline{p}_{\widetilde{E}}\left(\widetilde{u}_g(\theta_{a,\xi})\right) &= \widetilde{p}_{L(E,A)}\left(\widetilde{u}_g(\theta_{a,\xi})\right) = \widetilde{p}_{L(E,A)}\left(\theta_{\alpha_g^u(a),u_g(\xi)}\right) \\ &= \sup\{p\left(\alpha_g^u\left(a\right)\left\langle u_g\left(\xi\right),u_g\left(\zeta\right)\right\rangle\right);\overline{p}_E(u_g\left(\zeta\right))\leq 1\} \\ &= \widetilde{r}_{L(E,A)}\left(\theta_{a,\xi}\right) = \overline{r}_{\widetilde{E}}\left(\theta_{a,\xi}\right)\end{aligned}$$

where $r = p\circ\alpha_g^u \in S(A)$, for all $a\in A$ and for all $\xi\in E$, $\widetilde{u}_g$ extends to a continuous linear map, denoted also by $\widetilde{u}_g$, from $K(E,A)$ to $K(E,A)$. From

$$\begin{aligned}\left\langle\widetilde{u}_g(\theta_{a,\xi}),\widetilde{u}_g(\theta_{b,\eta})\right\rangle &= \left\langle\theta_{\alpha_g^u(a),u_g(\xi)},\theta_{\alpha_g^u(b),u_g(\eta)}\right\rangle \\ &= \theta_{u_g(\xi),\alpha_g^u(a)}\circ\theta_{\alpha_g^u(b),u_g(\eta)} \\ &= \theta_{u_g(\xi a^*),u_g(\eta b^*)} = \beta_g^u\left(\theta_{\xi a^*\eta b^*}\right) \\ &= \beta_g^u\left(\left\langle\theta_{a,\xi},\theta_{b,\eta}\right\rangle\right)\end{aligned}$$

for all $\xi,\eta\in E$ and for all $a,b\in A$, and taking into account that $\widetilde{u}_g$ is continuous and $\Theta\left(E,A\right)$ is dense in $K(E,A)$, we conclude that $\widetilde{u}_g$ is a morphism of Hilbert modules. Moreover, since $\widetilde{u}_g$ is invertible and $\left(\widetilde{u}_g\right)^{-1}=\widetilde{u}_{g^{-1}}$, $\widetilde{u}_g$ is an isomorphism of Hilbert modules. A simple calculation shows that the map $g\to\widetilde{u}_g$ is an action of G on $\widetilde{E}$.

Now, we suppose that $g\to u_g$ is a continuous inverse limit action. Then $u_g = \lim\limits_{\substack{\leftarrow\\ p\in S(A)}} u_g^p$ for each $g\in G$ and $g\to u_g^p$ is a continuous action of G on E_p for each $p\in S(A)$. For each $p\in S(A)$, the continuous action u^p of G on E_p induces a continuous action $\widetilde{u^p}$ of G on $\widetilde{E_p}$. It is not difficult to check that for each $g\in G$, $\left(\widetilde{u}_g^p\right)_p$ is an inverse system of isomorphisms of Hilbert modules and then $g\to \lim\limits_{\substack{\leftarrow\\ p\in S(A)}}\widetilde{u}_g^p$ is a continuous inverse limit action of G on $\lim\limits_{\substack{\leftarrow\\ p\in S(A)}}\widetilde{E_p}$. Since the Hilbert A-modules $\widetilde{E}$ and $\lim\limits_{\substack{\leftarrow\\ p\in S(A)}}\widetilde{E_p}$ are unitarily equivalent [6, pp. 805-806], the action $g\to\lim\limits_{\substack{\leftarrow\\ p\in S(A)}}\widetilde{u}_g^p$ of G on $\lim\limits_{\substack{\leftarrow\\ p\in S(A)}}\widetilde{E_p}$ can be identified with an action of G on $\widetilde{E}$. Moreover,

$$\left(\lim_{\substack{\leftarrow\\ p\in S(A)}}\widetilde{u}_g^p\right)\left(\theta_{a,\xi}\right)) = \theta_{\alpha_g^u(a),u_g(\xi)}$$

for all $a\in A$ and for all $\xi\in E$. □

Remark 4.9. *Let G be a locally compact group and let $g\to u_g$ be an action of G on a full Hilbert A-module E.*

1. *The action of G on $K(E)$ induced by $\widetilde{u}$ coincides with the action of G on $K(E)$ induced by u.*
2. *Since $\varphi: K(\widetilde{E})\to A$ defined by*

$$\varphi\left(\theta_{\theta_{a,\xi},\theta_{b,\eta}}\right) = \left\langle\xi a^*,\eta b^*\right\rangle$$

is an isomorphism of locally C^*-algebras, and since

$$\begin{aligned}\left(\alpha_g^u \circ \varphi\right)\left(\theta_{\theta_{a,\xi},\theta_{b,\eta}}\right) &= \alpha_g^u\left(\langle \xi a^*, \eta b^*\rangle\right)\\ &= \left\langle u_g\left(\xi\right)\alpha_g^u\left(a\right)^*, u_g\left(\eta\right)\alpha_g^u\left(b\right)^*\right\rangle\\ &= \varphi\left(\theta_{\theta\alpha_g^u(a),u_g(\xi),\theta\alpha_g^u(b),u_g(\eta)}\right)\\ &= \varphi\left(\theta_{\widetilde{u}_g(\theta_{a,\xi}),\widetilde{u}_g(\theta_{b,\xi})}\right)\\ &= \left(\varphi\circ\beta_g^{\widetilde{u}}\right)\left(\theta_{\theta_{a,\xi},\theta_{b,\eta}}\right)\end{aligned}$$

for all $a, b \in A$, for all $\xi, \eta \in E$ and for all $g \in G$, the action of G on A induced by u is conjugate with the action of G on $K(\widetilde{E})$ induced by $\widetilde{u}$.

Let E be a Hilbert A-module, let F be a Hilbert B-module and let $\Phi : A \to L(F)$ be a non-degenerate representation of A on F. The tensor product $E \otimes_A F$ of E and F over A becomes a pre-Hilbert B-module with the action of B on $E \otimes_A F$ defined by

$$(\xi \otimes_A \eta)\, b = \xi \otimes_A \eta b$$

and the inner product defined by

$$\langle \xi_1 \otimes_A \eta_1, \xi_2 \otimes_A \eta_2\rangle_\Phi = \langle \eta_1, \Phi\left(\langle \xi_1, \xi_2\rangle\right)\eta_2\rangle .$$

The completion of $E \otimes_A F$ with respect to the topology determined by the inner product is called the interior tensor product of the Hilbert modules E and F using Φ and it is denoted by $E \otimes_\Phi F$ [11, Proposition 4.1]. An element in $E \otimes_\Phi F$ is denoted by $\xi \otimes_\Phi \eta$.

Definition 4.10. *Let $g \to \alpha_g$ be an action of G on A and let $\Phi : A \to L(F)$ be a non-degenerate representation of A on a Hilbert B-module F. An action $g \to v_g$ of G on F is Φ covariant relative to α if*

$$v_g \Phi\left(a\right) v_{g^{-1}} = \Phi\left(\alpha_g(a)\right)$$

for all $a \in A$ and for all $g \in G$.

Proposition 4.11. *Let G be a locally compact group, let E be a full Hilbert A-module, let F be a full Hilbert B-module and let $\Phi : A \to L(F)$ be a non-degenerate representation of A on F. If $g \to u_g$ is an action of G on E and $g \to v_g$ is an action of G on F which is Φ covariant relative to α^u (the action of G on A induced by u), then there is a unique action $g \to w_g^{u,v}$ of G on $E \otimes_\Phi F$ such that*

$$w_g^{u,v}\left(\xi \otimes_\Phi \eta\right) = u_g\left(\xi\right) \otimes_\Phi v_g\left(\eta\right)$$

for all $\xi \in E$, for all $\eta \in F$ and for all $g \in G$. Moreover, if $g \to u_g$ is a continuous action of G on E and $g \to v_g$ is a continuous inverse limit action of G on F, then $g \to w_g^{u,v}$ is a continuous inverse limit action of G on $E \otimes_\Phi F$.

Proof. Let $g \in G$. Since

$$u_g(\xi a) \otimes_\Phi v_g(\eta) = u_g(\xi) \otimes_\Phi v_g \Phi(a)(\eta)$$

for all $\xi \in E$, for all $\eta \in F$ and for all $a \in A$, and since

$$\begin{aligned} \overline{q}_{E\otimes_\Phi F}\left(u_g(\xi) \otimes_\Phi v_g(\eta)\right)^2 &= q\left(\left\langle v_g(\eta), \Phi\left(\alpha_g^u(\langle \xi, \xi\rangle)\right) v_g(\eta)\right\rangle\right) \\ &= q\left(\langle v_g(\eta), v_g(\Phi(\langle \xi, \xi\rangle)(\eta))\rangle\right) \\ &= q\left(\alpha_g^v(\langle \eta, \Phi(\langle \xi, \xi\rangle)(\eta)\rangle)\right) \\ &= \left(q \circ \alpha_g^v\right)\left(\langle \xi \otimes_\Phi \eta, \xi \otimes_\Phi \eta\rangle_\Phi\right) \\ &= \overline{r}_{E\otimes_\Phi F}(\xi \otimes_\Phi \eta)^2 \end{aligned}$$

where $r = q \circ \alpha_g^v \in S(B)$, for all $\xi \in E$ and for all $\eta \in F$, there is a continuous linear map $w_g^{u,v} : E \otimes_\Phi F \to E \otimes_\Phi F$ such that

$$w_g^{u,v}(\xi \otimes_\Phi \eta) = u_g(\xi) \otimes_\Phi v_g(\eta).$$

From

$$\left\langle w_g^{u,v}(\xi_1 \otimes_\Phi \eta_1), w_g^{u,v}(\xi_2 \otimes_\Phi \eta_2)\right\rangle_\Phi = \alpha_g^v\left(\langle \xi_1 \otimes_\Phi \eta_1, \xi_2 \otimes_\Phi \eta_2\rangle_\Phi\right)$$

for all $\xi_1, \xi_2 \in E$ and for all $\eta_1, \eta_2 \in F$ and taking into account that $w_g^{u,v} \circ w_{g^{-1}}^{u,v} = w_{g-1}^{u,v} \circ w_g^{u,v} = \mathrm{id}_{\ E\otimes_\Phi F}$, we conclude that $w_g^{u,v} \in \mathrm{Aut}(E \otimes_\Phi F)$. Since $\{\xi \otimes_\Phi \eta; \xi \in E, \eta \in F\}$ generates $E \otimes_\Phi F$, $w_g^{u,v}$ is the unique linear map from $E \otimes_\Phi F$ to $E \otimes_\Phi F$ such that

$$w_g^{u,v}(\xi \otimes_\Phi \eta) = u_g(\xi) \otimes_\Phi v_g(\eta)$$

for all $\xi \in E$, for all $\eta \in F$ and for all $g \in G$. It is not difficult to check that $g \to w_g^{u,v}$ is an action of G on $E \otimes_\Phi F$.

To show that the action $w^{u,v}$ of G on $E \otimes_\Phi F$ is a continuous inverse limit action if u is continuous and v is a continuous inverse limit action, we partition the proof into two steps.

Step 1. We suppose that B is a C^*-algebra.

Let $x_0 \in E \otimes_\Phi F$ and $\varepsilon > 0$. Then there is $\xi_0 \in E$ and $\eta_0 \in F$ such that

$$\|x_0 - \xi_0 \otimes_\Phi \eta_0\| \le \varepsilon/6.$$

Since Φ is a representation of A on F there is $p \in S(A)$ such that

$$\|\Phi(a)\| \le p(a)$$

for all $a \in A$, and since $g \to u_g$ and $g \to v_g$ are continuous, there is a neighborhood U_0 of g_0 such that

$$\overline{p}_E(u_g(\xi_0) - \xi_0) \le \min\{\sqrt{\varepsilon/6}, \varepsilon/(6\|\eta_0\|)\}$$

and

$$\|v_g(\eta_0) - \eta_0\| \le \min\{\sqrt{\varepsilon/6}, \varepsilon/(6\overline{p}_E(\xi_0))\}$$

for all $g \in U_0$. Then, since

$$\begin{aligned} \|\xi \otimes_\Phi \eta\|^2 &= \|\langle \xi \otimes_\Phi \eta, \xi \otimes_\Phi \eta\rangle_\Phi\| = \|\langle \eta, \Phi(\langle \xi, \xi\rangle)\eta\rangle\| \\ &\leq \|\eta\| \|\Phi(\langle \xi, \xi\rangle)\eta\| \leq \|\eta\|^2 \|\Phi(\langle \xi, \xi\rangle)\| \\ &\leq \|\eta\|^2 p(\langle \xi, \xi\rangle) = \|\eta\|^2 \overline{p}_E(\xi)^2 \end{aligned}$$

for all $\xi \in E$ and for all $\eta \in F$, we have

$$\begin{aligned} \left\|w_g^{u,v}(x_0) - x_0\right\| &\leq \left\|w_g^{u,v}(x_0 - \xi_0 \otimes_\Phi \eta_0)\right\| + \left\|w_g^{u,v}(\xi_0 \otimes_\Phi \eta_0) - \xi_0 \otimes_\Phi \eta_0\right\| \\ &\quad + \|x_0 - \xi_0 \otimes_\Phi \eta_0\| \\ &\leq 2\|x_0 - \xi_0 \otimes_\Phi \eta_0\| + \|u_g(\xi_0) \otimes_\Phi v_g(\eta_0) - \xi_0 \otimes_\Phi \eta_0\| \\ &\leq \varepsilon/3 + \|(u_g(\xi_0) - \xi_0) \otimes_\Phi (v_g(\eta_0) - \eta_0)\| \\ &\quad + \|\xi_0 \otimes_\Phi (v_g(\eta_0) - \eta_0)\| + \|(u_g(\xi_0) - \xi_0) \otimes_\Phi \eta_0\| \\ &\leq \varepsilon/3 + \|v_g(\eta_0) - \eta_0\| \overline{p}_E(u_g(\xi_0) - \xi_0) \\ &\quad + \|v_g(\eta_0) - \eta_0\| \overline{p}_E(\xi_0) + \|\eta_0\| \overline{p}_E(u_g(\xi_0) - \xi_0) \\ &\leq \varepsilon \end{aligned}$$

for all $g \in U_0$. Therefore the action $w^{u,v}$ of G on $E \otimes_\Phi F$ is continuous.

Step 2. The general case when B is a locally C^*-algebra. By [11, Proposition 4.2], $E \otimes_\Phi F$ can be identified with $\lim\limits_{\substack{\leftarrow \\ q \in S(B)}} E \otimes_{\Phi_q} F_q$, where $\Phi_q = \left(\pi_q^B\right)_* \circ \Phi$ for each $q \in S(B)$.

Let $q \in S(B)$. From

$$\begin{aligned} v_g^q \Phi_q(a) v_{g^{-1}}^q \left(\sigma_q^F(\eta)\right) &= \left(v_g^q \Phi_q(a)\right)\left(\sigma_q^F\left(v_{g^{-1}}(\eta)\right)\right) \\ &= v_g^q\left(\sigma_q^F\left(\Phi(a) v_{g^{-1}}(\eta)\right)\right) \\ &= \sigma_q^F\left(v_g^q \Phi(a) v_{g^{-1}}(\eta)\right) = \sigma_q^F\left(\Phi\left(\alpha_g^u(a)\right)(\eta)\right) \\ &= \Phi_q\left(\alpha_g^u(a)\right)\left(\sigma_q^F(\eta)\right) \end{aligned}$$

for all $\eta \in F$, for all $a \in A$ and for all $g \in G$, we conclude that the action $g \to v_g^q$ of G on F_q is Φ_q covariant relative to α^u. Then, by the first step of this prof, $g \to w_g^{u,v^q}$ is a continuous action of G on $E \otimes_{\Phi_q} F_q$. It is not difficult to check that for each $g \in G$, $\left(w_g^{u,v^q}\right)_q$ is an inverse system of isomorphisms of Hilbert C^*-modules, and the map $g \to \lim\limits_{\substack{\leftarrow \\ q \in S(B)}} w_g^{u,v^q}$ is a continuous inverse limit action of G on $\lim\limits_{\substack{\leftarrow \\ q \in S(B)}} E \otimes_{\Phi_q} F_q$. Moreover, this action can be identified with the action $g \to w_g^{u,v}$ of G on $E \otimes_\Phi F$. $\square$

Remark 4.12. (1) *The action of G on B induced by the action $w^{u,v}$ of G on $E \otimes_\Phi F$ coincides with the action of G on B induced by v.*

(2) *Suppose that $\Phi : A \to K(E)$ is an isomorphism of locally C^*-algebras. Then $\Phi_* : K(E) \to K(E \otimes_\Phi F)$ defined by $\Phi_*(T)(\xi \otimes_\Phi \eta) = T(\xi) \otimes_\Phi \eta$ is an*

isomorphism of locally C^*-algebras [11, Proposition 4.4 and Corollary 4.6]. *In this case, the action of G on $K(E\otimes_\Phi F)$ induced by $w^{u,v}$ is conjugate with the action of G on $K(E)$ induced by u. Indeed, we have*

$$\begin{aligned}\left(\beta_g^{w^{u,v}}\circ\Phi_*\right)(T)\left(\xi\otimes_\Phi\eta\right) &= \left(w_g^{u,w}\Phi_*(T)\,w_{g^{-1}}^{u,w}\right)\left(\xi\otimes_\Phi\eta\right)\\ &= \left(w_g^{u,w}\Phi_*(T)\right)\left(u_{g^{-1}}(\xi)\otimes_\Phi v_{g^{-1}}(\eta)\right)\\ &= w_g^{u,w}\left(T\left(u_{g^{-1}}(\xi)\right)\otimes_\Phi v_{g^{-1}}(\eta)\right)\\ &= u_gTu_{g^{-1}}(\xi)\otimes_\Phi\eta\\ &= \beta_g^u(T)(\xi)\otimes_\Phi\eta\\ &= \Phi_*\left(\beta_g^u(T)\right)\left(\xi\otimes_\Phi\eta\right)\\ &= \left(\Phi_*\circ\beta_g^u\right)(T)\left(\xi\otimes_\Phi\eta\right)\end{aligned}$$

for all $\xi\in E$ and for all $\eta\in F$.

Proposition 4.13. *Morita equivalence of group actions on locally C^*-algebras is an equivalence relation.*

Proof. Let α be an action of G on the locally C^*-algebra A. Clearly, we can regard α as an action of G on the Hilbert A-module A. Since the Hilbert A-module A is full and since $K(A)$ is isomorphic with A, $\alpha\backsim_{A,\alpha}\alpha$. Therefore the relation is reflexive.

From Proposition 4.8 and Remark 4.9, we conclude that the relation is symmetric.

To show that the relation is transitive, let α,β,γ be three actions of G on the locally C^*-algebras A, B and C such that $\alpha\backsim_{E,u}\beta$ and $\beta\backsim_{F,v}\gamma$. By [6, the proof of Proposition 4.4], $F\otimes_i E$, where i is the embedding of $K(E)$ in $L(E)$, is a full Hilbert A-module such that the locally C^*-algebras $K(E)$ and $K(F\otimes_i E)$ are isomorphic. Since

$$i\left(\beta_g^u\left(\theta_{\xi,\eta}\right)\right)=u_gi\left(\theta_{\xi,\eta}\right)u_{g^{-1}}$$

for all $g\in G$ and for all $\xi,\eta\in E$, the action u is covariant with respect to i relative to β^u. Then, from Proposition 4.11 and Remark 4.12, we conclude that the pair $(F\otimes_i E,w^{u,v})$ implements a Morita equivalence between α and γ. □

In the same way as in the proof of Proposition 4.13 and using Remark 4.7, we obtain the following proposition.

Proposition 4.14. *Strong Morita equivalence of continuous inverse limit group actions on locally C^*-algebras is an equivalence relation.*

5. Crossed products of locally C^*-algebras associated with strong Morita equivalent actions

Let G be a locally compact group and let E be a Hilbert module over a locally C^*-algebra A.

Let $C_c(G,E)$ be the vector space of all continuous functions from G to E with compact support.

Remark 5.1. *If $\widehat{\xi},\widehat{\eta}\in C_c(G,E)$, then the function $\left\langle \widehat{\xi},\widehat{\eta}\right\rangle : G\to A$ defined by $\left\langle \widehat{\xi},\widehat{\eta}\right\rangle(s)=\left\langle \widehat{\xi}(s),\widehat{\eta}(s)\right\rangle$ is with compact support.*

Lemma 5.2. *Let $\widehat{\xi}\in C_c(G,E)$. Then there is a unique element $\int\limits_G \widehat{\xi}(s)ds\in E$, such that*

$$\left\langle \int\limits_G \widehat{\xi}(s)ds,\eta\right\rangle=\int\limits_G \left\langle \widehat{\xi}(s),\eta\right\rangle ds$$

for all $\eta\in E$. Moreover,

(1) $\overline{p}_E\left(\int\limits_G \widehat{\xi}(s)ds\right)\le M_{\widehat{\xi}}\sup\{\overline{p}_E\left(\widehat{\xi}(s)\right);s\in G\}$ for some positive number $M_{\widehat{\xi}}$;

(2) $\left(\int\limits_G \widehat{\xi}(s)ds\right)a=\int\limits_G \widehat{\xi}(s)ads$ for all $a\in A$;

(3) $T\left(\int\limits_G \widehat{\xi}(s)ds\right)=\int\limits_G T\left(\widehat{\xi}(s)\right)ds$ for all $T\in L(E,F)$.

Proof. Consider the map $T\left(\widehat{\xi}\right):E\to A$ defined by

$$T\left(\widehat{\xi}\right)(\eta)=\int\limits_G \left\langle \widehat{\xi}\left(s\right),\eta\right\rangle ds.$$

By Lemma 2.2, $T\left(\widehat{\xi}\right)$ is a module morphism. Let $p\in S(A)$. Then

$$\begin{aligned}
\widetilde{p}_{L(E,A)}\left(T\left(\widehat{\xi}\right)\right) &= \sup\{p\left(T\left(\widehat{\xi}\right)(\eta)\right);\overline{p}_E(\eta)\le 1\}\\
&= \sup\{p\left(\int\limits_G \left\langle \widehat{\xi}\left(s\right),\eta\right\rangle ds\right);\overline{p}_E(\eta)\le 1\}\\
&\le M_{\widehat{\xi}}\sup\{\sup\{p\left(\left\langle \widehat{\xi}\left(s\right),\eta\right\rangle\right);s\in G\};\overline{p}_E(\eta)\le 1\}\\
&\le M_{\widehat{\xi}}\sup\{\overline{p}_E\left(\widehat{\xi}\left(s\right)\right);s\in G\}
\end{aligned}$$

where $M_{\widehat{\xi}}=\int_{G_{\widehat{\xi}}}ds$, $G_{\widehat{\xi}}=\operatorname{supp}\widehat{\xi}$.

If $\widehat{\xi}=\xi\otimes f$, $\xi\in E$ and $f\in C_c(G)$, it is not difficult to check that $T\left(\widehat{\xi}\right)=T_\zeta$, where $\zeta=\left(\int_G f(s)ds\right)\xi$ and $T_\zeta\left(\eta\right)=\langle\zeta,\eta\rangle$ for all $\eta\in E$. Therefore $T\left(\widehat{\xi}\right)\in K(E,A)$.

Now suppose that $\widehat{\xi}\in C_c(G,E)$. For $\varepsilon>0$ and $p\in S(A)$, there exist $\xi_i\in E$ and $f_i\in C_c(G)$, $i=1,2,...,n$, such that

$$\sup\{\overline{p}_E\left(\widehat{\xi}\left(s\right)-\sum_{i=1}^n f_i(s)\xi_i\right),s\in G\}\le\varepsilon/M_{\widehat{\xi}}.$$

Then

$$\begin{aligned}\widetilde{p}_{L(E,A)}\left(T\left(\widehat{\xi}\right)-T\left(\sum_{i=1}^{n}\xi_i\otimes f_i\right)\right) &\leq M_{\widehat{\xi}}\sup\{\overline{p}_E\left(\widehat{\xi}\,(s)-\sum_{i=1}^{n}f_i(s)\xi_i\right), s\in G\} \\ &\leq \varepsilon.\end{aligned}$$

From these facts we conclude that $T\left(\widehat{\xi}\right)\in K(E,A)$. Therefore there is a unique element $\int\limits_G \widehat{\xi}(s)ds\in E$ such that

$$T\left(\widehat{\xi}\right)(\eta)=\left\langle\int\limits_G\widehat{\xi}(s)ds,\eta\right\rangle$$

and so

$$\left\langle\int\limits_G\widehat{\xi}(s)ds,\eta\right\rangle=\int\limits_G\left\langle\widehat{\xi}(s),\eta\right\rangle ds$$

for all $\eta\in E$. Moreover, $\widetilde{p}_{L(E,A)}\left(T\left(\widehat{\xi}\right)\right)=\overline{p}_E\left(\int\limits_G\widehat{\xi}(s)ds\right)$ [10, Lemma 2.1.3 and Corollary 1.2.3], and thus the relation (1) is verified.

Let $a\in A$. Then

$$\begin{aligned}\left\langle\int\limits_G\widehat{\xi}(s)ads,\eta\right\rangle &= \int\limits_G\left\langle\widehat{\xi}(s)a,\eta\right\rangle ds=a^*\int\limits_G\left\langle\widehat{\xi}(s),\eta\right\rangle ds \\ &= a^*\left\langle\int\limits_G\widehat{\xi}(s)ds,\eta\right\rangle=\left\langle\left(\int\limits_G\widehat{\xi}(s)ds\right)a,\eta\right\rangle\end{aligned}$$

for all $\eta\in E$. This implies that

$$\int\limits_G\widehat{\xi}(s)ads=\left(\int\limits_G\widehat{\xi}(s)ds\right)a$$

and thus the relation (2) is verified. Let $T\in L(E,F)$. From

$$\begin{aligned}\left\langle T\left(\int\limits_G\widehat{\xi}(s)ds\right),\eta\right\rangle &= \left\langle\int\limits_G\widehat{\xi}(s)ds,T^*(\eta)\right\rangle=\int\limits_G\left\langle\widehat{\xi}(s),T^*(\eta)\right\rangle ds \\ &= \int\limits_G\left\langle T\left(\widehat{\xi}(s)\right),\eta)\right\rangle ds=\left\langle\int\limits_G T\left(\widehat{\xi}(s)\right)ds,\eta)\right\rangle\end{aligned}$$

we conclude that $T\left(\int\limits_G\widehat{\xi}(s)ds\right)=\int\limits_G T\left(\widehat{\xi}(s)\right)ds$, and then the relation (3) is verified too. $\square$

Let $g\to u_g$ be a continuous inverse limit action of G on a full Hilbert A-module E. We can suppose that $u_g=\lim\limits_{\substack{\leftarrow\\ p\in S(A)}}u_g^p$ for each $g\in G$, where $g\to u_g^p$, $p\in S(A)$ are continuous actions of G on E_p.

Let $\widehat{\xi} \in C_c(G,E)$ and $f \in C_c(G,A)$. It is not difficult to check that the function

$$(s,t) \rightarrow \widehat{\xi}\,(s)\,\alpha_s^u\left(f\left(s^{-1}t\right)\right)$$

from $G \times G$ to E is continuous with compact support and the formula

$$t \rightarrow \left(\widehat{\xi} \cdot f\right)(t) = \int\limits_G \widehat{\xi}\,(s)\,\alpha_s^u\left(f\left(s^{-1}t\right)\right) ds,\ t \in G$$

defines an element $\widehat{\xi} \cdot f \in C_c(G,E)$. Thus, we have defined a map from $C_c(G,A) \times C_c(G,E) \rightarrow C_c(G,E)$ by

$$\left(\widehat{\xi}, f\right) \mapsto \widehat{\xi} \cdot f.$$

It is not difficult to check that this map is $\mathbb{C}$ -linear with respect to its variables. Let $\widehat{\xi} \in C_c(G,E)$ and $f,h \in C_c(G,A)$. From

$$\begin{aligned}
\left(\widehat{\xi} \cdot (f \times h)\right)(t) &= \int\limits_G \widehat{\xi}\,(s)\,\alpha_s^u\left((f \times h)\left(s^{-1}t\right)\right) ds \\
&= \int\limits_G \widehat{\xi}\,(s)\,\alpha_s^u\left(\int\limits_G f\,(r)\,\alpha_r^u\left(h\left(r^{-1}s^{-1}t\right)\right) dr\right) ds \\
&= \int\limits_G \widehat{\xi}\,(s)\left(\int\limits_G \alpha_s^u\left(f\left(s^{-1}g\right)\right)\alpha_g^u\left(h\left(g^{-1}t\right)\right) dg\right) ds \\
&= \int\limits_G \left(\int\limits_G \widehat{\xi}\,(s)\,\alpha_s^u\left(f\left(s^{-1}g\right)\right)\alpha_r^u\left(h\left(g^{-1}t\right)\right) dg\right) ds
\end{aligned}$$

and

$$\begin{aligned}
\left(\left(\widehat{\xi} \cdot f\right) \cdot h\right)(t) &= \int\limits_G \left(\widehat{\xi} \cdot f\right)(s)\,\alpha_s^u\left(h\left(s^{-1}t\right)\right) ds \\
&= \int\limits_G \left(\int\limits_G \widehat{\xi}\,(r)\,\alpha_r^u\left(f\left(r^{-1}s\right)\right) dr\right)\alpha_s^u\left(h\left(s^{-1}t\right)\right) ds \\
&= \int\limits_G \left(\int\limits_G \widehat{\xi}\,(r)\,\alpha_r^u\left(f\left(r^{-1}s\right)\right)\alpha_s^u\left(h\left(s^{-1}t\right)\right) dr\right) ds \\
&= \int\limits_G \left(\int\limits_G \widehat{\xi}\,(r)\,\alpha_r^u\left(f\left(r^{-1}s\right)\right)\alpha_s^u\left(h\left(s^{-1}t\right)\right) ds\right) dr
\end{aligned}$$

for all $t \in G$, we conclude that

$$\widehat{\xi} \cdot (f \times h) = \left(\widehat{\xi} \cdot f\right) \cdot h.$$

Therefore $C_c(G,E)$ has a structure of right $C_c(G,A)$-module.

Let $\widehat{\xi},\widehat{\eta}\in C_c(G,E)$. It is not difficult to check that the function

$$(s,t)\to\alpha^u_{s^{-1}}\left(\left\langle\widehat{\xi}\left(s\right),\widehat{\eta}\left(st\right)\right\rangle\right)$$

from $G\times G$ to A is continuous with compact support and the formula

$$t\to\int\limits_G\alpha^u_{s^{-1}}\left(\left\langle\widehat{\xi}\left(s\right),\widehat{\eta}\left(st\right)\right\rangle\right)ds$$

defines an element in $C_c(G,A)$. Thus, we have defined a $C_c(G,A)$-valued inner product $\langle\cdot,\cdot\rangle_{C_c(G,E)}:C_c(G,E)\times C_c(G,E)\to C_c(G,A)$ by

$$\left\langle\widehat{\xi},\widehat{\eta}\right\rangle_{C_c(G,E)}(t)=\int\limits_G\alpha^u_{s^{-1}}\left(\left\langle\widehat{\xi}\left(s\right),\widehat{\eta}\left(st\right)\right\rangle\right)ds.$$

It is not difficult to check that the inner product defined above is $\mathbb{C}$-linear with respect to its second variable.

Let $f\in C_c(G,A)$, $\widehat{\xi},\widehat{\eta}\in C_c(G,E)$. From

$$\begin{aligned}
\left\langle\widehat{\xi},\widehat{\eta}\cdot f\right\rangle_{C_c(G,E)}(t) &= \int\limits_G\alpha^u_{s^{-1}}\left(\left\langle\widehat{\xi}\left(s\right),\left(\widehat{\eta}\cdot f\right)\left(st\right)\right\rangle\right)ds\\
&= \int\limits_G\alpha^u_{s^{-1}}\left(\left\langle\widehat{\xi}\left(s\right),\int\limits_G\widehat{\eta}\left(g\right)\alpha^u_g\left(f\left(g^{-1}st\right)\right)dg\right\rangle\right)ds\\
&= \int\limits_G\alpha^u_{s^{-1}}\left(\int\limits_G\left(\left\langle\widehat{\xi}\left(s\right),\widehat{\eta}\left(g\right)\alpha^u_g\left(f\left(g^{-1}st\right)\right)\right\rangle\right)dg\right)ds\\
&= \int\limits_G\left(\int\limits_G\alpha^u_{s^{-1}}\left(\left\langle\widehat{\xi}\left(s\right),\widehat{\eta}\left(g\right)\alpha^u_g\left(f\left(g^{-1}st\right)\right)\right\rangle\right)dg\right)ds\\
&= \int\limits_G\left(\int\limits_G\alpha^u_{s^{-1}}\left(\left\langle\widehat{\xi}\left(s\right),\widehat{\eta}\left(g\right)\right\rangle\right)\alpha^u_{s^{-1}g}\left(f\left(g^{-1}st\right)\right)dg\right)ds
\end{aligned}$$

and

$$\begin{aligned}
\left(\left\langle \widehat{\xi}, \widehat{\eta}\right\rangle_{C_c(G,E)} \times f\right)(t) &= \int\limits_G \left\langle \widehat{\xi}, \widehat{\eta}\right\rangle_{C_c(G,A)}(s)\alpha_s^u\left(f\left(s^{-1}t\right)\right) ds \\
&= \int\limits_G \left(\int\limits_G \alpha_{r^{-1}}^u\left(\left\langle \widehat{\xi}\left(r\right), \widehat{\eta}\left(rs\right)\right\rangle\right) dr\right) \alpha_s^u\left(f\left(s^{-1}t\right)\right) ds \\
&= \int\limits_G \left(\int\limits_G \alpha_{r^{-1}}^u\left(\left\langle \widehat{\xi}\left(r\right), \widehat{\eta}\left(rs\right)\right\rangle\right) \alpha_s^u\left(f\left(s^{-1}t\right)\right) dr\right) ds \\
&\quad \text{Fubini's Theorem} \\
&= \int\limits_G \left(\int\limits_G \alpha_{r^{-1}}^u\left(\left\langle \widehat{\xi}\left(r\right), \widehat{\eta}\left(rs\right)\right\rangle\right) \alpha_s^u\left(f\left(s^{-1}t\right)\right) ds\right) dr \\
&= \int\limits_G \left(\int\limits_G \alpha_{r^{-1}}^u\left(\left\langle \widehat{\xi}\left(r\right), \widehat{\eta}\left(g\right)\right\rangle\right) \alpha_{r^{-1}g}^u\left(f\left(g^{-1}rt\right)\right) dg\right) dr
\end{aligned}$$

for all $t \in G$, we deduce that

$$\left\langle \widehat{\xi}, \widehat{\eta}\cdot f\right\rangle_{C_c(G,E)} = \left\langle \widehat{\xi}, \widehat{\eta}\right\rangle_{C_c(G,E)} \times f.$$

From

$$\begin{aligned}
\left(\left\langle \widehat{\xi}, \widehat{\eta}\right\rangle_{C_c(G,E)}\right)^{\#}(t) &= \Delta\left(t\right)^{-1}\alpha_t^u\left(\left(\left\langle \widehat{\xi}, \widehat{\eta}\right\rangle_{C_c(G,E)}\left(t^{-1}\right)\right)^*\right) \\
&= \Delta\left(t\right)^{-1}\alpha_t^u\left(\left(\int\limits_G \alpha_{s^{-1}}^u\left(\left\langle \widehat{\xi}\left(s\right), \widehat{\eta}\left(st^{-1}\right)\right\rangle\right) ds\right)^*\right) \\
&= \Delta\left(t\right)^{-1}\alpha_t^u\left(\int\limits_G \alpha_{s^{-1}}^u\left(\left\langle \widehat{\eta}\left(st^{-1}\right), \widehat{\xi}\left(s\right)\right\rangle\right) ds\right) \\
&= \Delta\left(t\right)^{-1}\int\limits_G \alpha_{ts^{-1}}^u\left(\left\langle \widehat{\eta}\left(st^{-1}\right), \widehat{\xi}\left(s\right)\right\rangle\right) ds \\
&= \Delta\left(t\right)^{-1}\int\limits_G \alpha_{g^{-1}}^u\left(\left\langle \widehat{\eta}\left(g\right), \widehat{\xi}\left(gt\right)\right\rangle\right) \Delta\left(t\right) dg \\
&= \left\langle \widehat{\eta}, \widehat{\xi}\right\rangle_{C_c(G,E)}(t)
\end{aligned}$$

for all $t \in G$, we deduce that

$$\left(\left\langle \widehat{\xi}, \widehat{\eta}\right\rangle_{C_c(G,E)}\right)^{\#} = \left\langle \widehat{\eta}, \widehat{\xi}\right\rangle_{C_c(G,E)}.$$

Let $\widehat{\xi} \in C_c(G,E)$ and $p \in S(A)$. Then

$$\begin{aligned}\left(\pi_p^A \circ \left\langle \widehat{\xi}, \widehat{\xi} \right\rangle_{C_c(G,E)}\right)(t) &= \pi_p^A \left(\int_G \alpha_{s^{-1}}^u \left(\left\langle \widehat{\xi}(s), \widehat{\xi}(st) \right\rangle\right) ds\right) \\ &= \int_G \pi_p^A \left(\alpha_{s^{-1}}^u \left(\left\langle \widehat{\xi}(s), \widehat{\xi}(st) \right\rangle\right)\right) ds \\ &= \int_G \alpha_{s^{-1}}^{u^p} \left(\left\langle \left(\sigma_p^E \circ \widehat{\xi}\right)(s), \left(\sigma_p^E \circ \widehat{\xi}\right)(st) \right\rangle\right) ds \\ &= \left(\left\langle \sigma_p^E \circ \widehat{\xi}, \sigma_p^E \circ \widehat{\xi} \right\rangle_{C_c(G,E_p)}\right)(t)\end{aligned}$$

for all $t \in G$. From this fact and [2, Remark pp. 300], we deduce that $\pi_p^A \circ \left\langle \widehat{\xi}, \widehat{\xi} \right\rangle_{C_c(G,E)}$ is a positive element in $C_c(G, A_p)$ for all $p \in S(A)$. Therefore $\left\langle \widehat{\xi}, \widehat{\xi} \right\rangle_{C_c(G,E)}$ is a positive element in $C_c(G, A)$.

Let $\widehat{\xi} \in C_c(G, A)$ such that $\left\langle \widehat{\xi}, \widehat{\xi} \right\rangle_{C_c(G,E)} = 0$. Then $\pi_p^A \circ \left\langle \widehat{\xi}, \widehat{\xi} \right\rangle_{C_c(G,E)} = 0$ for all $p \in S(A)$, and since $\pi_p^A \circ \left\langle \widehat{\xi}, \widehat{\xi} \right\rangle_{C_c(G,E)} = \left\langle \sigma_p^E \circ \widehat{\xi}, \sigma_p^E \circ \widehat{\xi} \right\rangle_{C_c(G,E_p)}$ for all $p \in S(A)$, we have $\left\langle \sigma_p^E \circ \widehat{\xi}, \sigma_p^E \circ \widehat{\xi} \right\rangle_{C_c(G,E_p)} = 0$ for all $p \in S(A)$. From this fact and [2, Remark pp. 300], we deduce that $\sigma_p^E \circ \widehat{\xi} = 0$ for all $p \in S(A)$ and so $\widehat{\xi} = 0$. Therefore $\left\langle \widehat{\xi}, \widehat{\xi} \right\rangle_{C_c(G,E)} = 0$ if and only of $\widehat{\xi} = 0$.

Thus we showed that $C_c(G, E)$ is a right $C_c(G, A)$-module equipped with a $C_c(G, A)$-valued inner product which is $\mathbb{C}$-and $C_c(G, A)$-linear in its second variable and verify the relations (1), (2) and (3) of Definition 2.1.

Let E^G be the completion of $C_c(G, E)$ with respect to the topology determined by the inner product if we consider on $C_c(G, A)$ the topology determined by the family of C^*-seminorms $\{n_p\}_{p \in S(A)}$. Then E^G is a Hilbert $G \times_{\alpha^u} A$-module [10, Remark 1.2.10].

Lemma 5.3. *Let $p \in S(A)$. Then the Hilbert $G \times_{\alpha^{u^p}} A_p$-modules $\left(E^G\right)_p$ and E_p^G are unitarily equivalent.*

Proof. Let $\widehat{\xi} \in C_c(G, E)$. Then

$$\begin{aligned}\overline{n_p}\left(\widehat{\xi}\right)^2 &= n_p\left(\left\langle \widehat{\xi}, \widehat{\xi} \right\rangle_{C_c(G,E)}\right) \\ &= \sup\{\left\| \varphi\left(\left\langle \widehat{\xi}, \widehat{\xi} \right\rangle_{C_c(G,E)}\right)\right\|; \varphi \in \mathcal{R}_p(G \times_{\alpha^u} A)\} \\ &= \sup\{\left\| \varphi_p\left(\left\langle \sigma_p^E \circ \widehat{\xi}, \sigma_p^E \circ \widehat{\xi} \right\rangle_{C_c(G,E_p)}\right)\right\|; \varphi_p \in \mathcal{R}\left((G \times_{\alpha^u} A)_p\right)\} \\ &= \left\| \sigma_p^E \circ \widehat{\xi} \right\|_{E_p^G}^2.\end{aligned}$$

Thus we can define a linear map $U_p : C_c(G,E)/\ker\left(\overline{n_p}|_{C_c(G,E)}\right) \to C_c(G,E_p)$ by

$$U_p\left(\widehat{\xi} + \ker\left(\overline{n_p}|_{C_c(G,E)}\right)\right) = \sigma_p^E \circ \widehat{\xi}.$$

Moreover,

$$\begin{aligned} & \left\langle U_p\left(\widehat{\xi} + \ker\left(\overline{n_p}|_{C_c(G,E)}\right)\right), U_p\left(\widehat{\xi} + \ker\left(\overline{n_p}|_{C_c(G,E)}\right)\right)\right\rangle_{C_c(G,E_p)} \\ = & \left\langle \sigma_p^E \circ \widehat{\xi}, \sigma_p^E \circ \widehat{\xi}\right\rangle_{C_c(G,E_p)} \\ = & \pi_p^A \circ \left\langle \widehat{\xi}, \widehat{\xi}\right\rangle_{C_c(G,E)} \\ = & \left\langle \widehat{\xi} + \ker\left(\overline{n_p}|_{C_c(G,E)}\right), \widehat{\xi} + \ker\left(\overline{n_p}|_{C_c(G,E)}\right)\right\rangle_{(C_c(G,E))_p} \end{aligned}$$

where $\left(C_c(G,E)\right)_p = C_c(G,E)/\ker\left(\overline{n_p}|_{C_c(G,E)}\right)$. For $\xi \in E$ and $f \in C_c(G)$, the map $s \to \widehat{\xi_f}$, where $\widehat{\xi_f}(s) = f(s)\xi$, defines an element in $C_c(G,E)$ and

$$U_p\left(\widehat{\xi_f} + \ker\left(\overline{n_p}|_{C_c(G,E)}\right)\right)(t) = f(t)\sigma_p^E(\xi) = \widehat{\sigma_p^E(\xi)}_f.$$

From these facts, [12, Theorem 3.5] and taking into account that the vector space $\left(C_c(G,E)\right)_p$ is dense in $\left(E^G\right)_p$ and $E_p \otimes_{\mathrm{alg}} C_c(G)$ is dense in E_p^G, we deduce that U_p extends to a unitary operator from $\left(E^G\right)_p$ to E_p^G. Therefore the Hilbert $G \times_{\alpha^{u^p}} A_p$-modules $\left(E^G\right)_p$ and E_p^G are unitarily equivalent. $\square$

Corollary 5.4. *The Hilbert $G \times_{\alpha^u} A$-modules E^G and $\lim\limits_{\substack{\leftarrow \\ p\in S(A)}} E_p^G$ are unitarily equivalent and the locally C^*-algebras $K\left(E^G\right)$ and $\lim\limits_{\substack{\leftarrow \\ p\in S(A)}} K(E_p^G)$ are isomorphic.*

Proof. If U_p is the unitary operator from $\left(E^G\right)_p$ to E_p^G constructed in the proof of Lemma 5.3, for each $p \in S(A)$, then $\left(U_p\right)_p$ is an inverse system of unitary operators and so the Hilbert $G \times_{\alpha^u} A$-module E^G and $\lim\limits_{\substack{\leftarrow \\ p\in S(A)}} E_p^G$ are unitarily equivalent. By [14, Proposition 4.7], the locally C^*-algebras $K\left(E^G\right)$ and $\lim\limits_{\substack{\leftarrow \\ p\in S(A)}} K(E_p^G)$ are isomorphic. $\square$

Definition 5.5. *([6]) Two locally C^*-algebras A and B are strongly Morita equivalent if there is a full Hilbert A-module E such that the locally C^*-algebras $K(E)$ and B are isomorphic.*

Theorem 5.6. *Let G be a locally compact group and let α and β be two continuous inverse limit actions of G on the locally C^*-algebras A and B. If the actions α and β are strongly Morita equivalent, then the crossed products $G \times_\alpha A$ and $G \times_\beta B$ are strongly Morita equivalent.*

Proof. Suppose that $\alpha \backsim_{E,u}^{s} \beta$. Then, by Proposition 4.4, the locally C^*-algebras $G \times_\alpha A$ and $G \times_{\alpha^u} A$ are isomorphic and so they are strongly Morita equivalent as well as the locally C^*-algebras $G \times_\beta B$ and $G \times_{\beta^u} K(E)$. To proof the theorem

it is sufficient to prove that the locally C^*-algebras $G\times_{\alpha^u} A$ and $G\times_{\beta^u} K(E)$ are strongly Morita equivalent.

For each $p\in S(A)$, a simple calculus shows that $\alpha^{u^p} \backsim^s_{E_p,u^p} \beta^{u^p}$ and by [2, Remark pp. 300], the Hilbert $G\times_{\alpha^{u^p}} A_p$-module E_p^G implements a strong Morita equivalence between $G\times_{\alpha^{u^p}} A_p$ and $G\times_{\beta^{u^p}} K(E_p)$. Moreover, the linear map $\Phi_p:\Theta(E_p^G)\to C_c(G,K(E_p))$ defined by

$$\Phi_p\left(\theta_{\widehat{\xi_p},\widehat{\eta_p}}\right)(t)=\int_G \theta_{\widehat{\xi_p}(s),\Delta(t^{-1}s)u_t^p\left(\widehat{\eta_p}(t^{-1}s)\right)}ds$$

extends to an isomorphism of C^*-algebras from $K(E_p^G)$ to $G\times_{\beta^{u^p}} K(E_p)$ [2, Remark pp. 300].

Let E^G be the Hilbert $G\times_{\alpha^u} A$-module constructed above. From Corollary 5.4 and taking into account that for each $p\in S(A)$, the Hilbert $G\times_{\alpha^{u^p}} A_p$-module E_p^G is full, we conclude that E^G is full. It is not difficult to check that

$$\pi_{pq}^{G\times_{\beta^u}K(E)}\circ\Phi_p=\Phi_q\circ\left(\pi_{pq}^{K(E^G)}\right)_*$$

for all $p,q\in S(A)$ with $p\geq q$, where $\{\pi_{pq}^{G\times_{\beta^u}K(E)}\}_{p,q\in S(A),p\geq q}$ and $\{\left(\pi_{pq}^{K(E^G)}\right)_*\}_{p,q\in S(A),p\geq q}$ are the connecting maps of the inverse systems of C^*-algebras $\{G\times_{\beta^{u^p}} K(E_p)\}_{p\in S(A)}$ respectively $\{K(E_p^G)\}_{p\in S(A)}$. Therefore $(\Phi_p)_p$ is an inverse system of C^*-isomorphisms, and then the locally C^*-algebras $\lim\limits_{\stackrel{\leftarrow}{p\in S(A)}} K(E_p^G)$ and $\lim\limits_{\stackrel{\leftarrow}{p\in S(A)}} G\times_{\beta^{u^p}} K(E_p)$ are isomorphic. From this fact, we deduce that the locally C^*-algebras $K\left(E^G\right)$ and $G\times_{\beta^u} K(E)$ are isomorphic and so the locally C^*-algebras $G\times_{\alpha^u} A$ and $G\times_{\beta^u} K(E)$ are strongly Morita equivalent. □

Corollary 5.7. *Let G be a compact group and let α and β be two continuous actions of G on the locally C^*-algebras A and B such that the maps $(g,a)\to\alpha_g(a)$ from $G\times A$ to A and $(g,b)\to\beta_g(b)$ from $G\times B$ to B are jointly continuous. If the actions α and β are strongly Morita equivalent, then the crossed products $G\times_\alpha A$ and $G\times_\beta B$ are strongly Morita equivalent.*

Proof. Since the group G is compact, the actions α and β of G on A and B are continuous inverse limit actions [15] and apply Theorem 5.6. □

References

[1] Bakic, D. and Guljas, B., *On a class of module maps of Hilbert C^*-modules,* Math. Commun. **7** (2002), no. 2, 177–192.

[2] Combes, F., *Crossed products and Morita equivalence*, Proc. London Math. Soc. **49**(1984),289-306.

[3] Curto E. R., Muhly.P.S., Williams P.D., *Cross products of strongly Morita equivalent C^*-algebras,* Proc. Amer. Math. Soc. **90**(1984), 4, 528,530.

[4] Fragoulopoulou, M., *Topological algebras with involution*, North-Holland Mathematics Studies, 200. Elsevier Science B.V., Amsterdam, 2005. xvi+495 pp. ISBN: 0-444-52025-2

[5] Inoue, A., *Inverse limit of C^*-algebras,* Mem. Faculty Sci. Kyushu Univ. Ser. A, **25**(1971), 197-235.

[6] Joiţa, M., *Morita equivalence for locally C^*-algebras*, Bull. London Math. Soc. **36**(2004), 802-810.

[7] Joiţa, M., *On the linking algebra of a Hilbert module and strong Morita equivalence of locally C^*-algebras*, Surveys in Mathematics and its Applications 1 (2006), 23-32, (Proceedings of Operator Models and Applications (OMA 2006), Timisoara , June 29-July 4, 2006).

[8] Joiţa, M., *Crossed products of locally C^*-algebras*, Rocky Mountain J. Math. (to appear), math.OA/0512128.

[9] Joiţa, M., *Group actions on Hilbert modules over locally C^*-algebras,* International Conference on Mathematics and Informatics, Bacau, Romania, September, 18-20, 2006.

[10] Joiţa, M., *Hilbert modules over locally C^*-algebras*, University of Bucharest Press, (2006), 150 pg. ISBN 973737128-3.

[11] Joiţa, M., *Tensor products of Hilbert modules over locally C^*-algebras*, Czech. Math. J., **54**(129) (2004), no. 3, 727-737.

[12] Lance, E. C., *Hilbert C^*-modules. A toolkit for operator algebraists,* London Mathematical Society Lecture Note Series 210, Cambridge University Press, Cambridge 1995.

[13] Mallios, A., *Topological algebras:* Selected Topics, North Holland, Amsterdam, 1986.

[14] Phillips, N.C.,*Inverse limits of C^*-algebras,* J. Operator Theory, **19** (1988), 159-195.

[15] Phillips, N.C., *Representable K-theory for σ-C^*-algebras*, K-Theory, **3** (1989), 5, 441–478.

[16] Raeburn, I. and Williams, P.D., *Morita equivalence and continuous-trace C^*-algebras,* Mathematical Surveys and Monographs, 60. American Mathematical Society, Providence, RI, 1998. xiv+327 pp. ISBN: 0-8218-0860-5.

[17] Williams, P.D., *Crossed products of C^*-algebras,* Mathematical Surveys and Monographs (**134**) 2007, American Mathematical Society, Providence, RI,; 528 pp; ISBN-10: 0-8218-4242-0.

Department of Mathematics, Faculty of Chemistry, University of Bucharest, Bd. Regina Elisabeta nr. 4-12, Bucharest, Romania

mjoita@fmi.unibuc.ro